Lecture 19
Solution of the Spherical-Geometry S_n Equations

1 Source Iteration

The spherical-geometry S_n equations are solved in a manner that is very similar to the solution of the slab-geometry equations. Source iteration is the basic iteration scheme, as it is in all S_n calculations. Expressed in terms of the continuum equation, this iteration takes the following form:

$$\frac{\mu}{r^2}\frac{\partial}{\partial r}r^2\psi^{\ell+1} + \frac{1}{r}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\psi^{\ell+1}\right] + \sigma_t\psi = \sum_{k=0}^{L}\frac{2k+1}{4\pi}(\sigma_k\phi_k^{\ell} + q_k)P_k(\mu)\,, \qquad (1)$$

where ℓ is the source iteration index. The only difference between source iteration in slab geometry and source iteration in spherical geometry is that the angles must be swept in a specific sequence in spherical geometry if the exact solution with a known total source is to be obtained with a single sweep. Specifically, the starting direction flux is first swept from right to left. This provides $\psi_{m=\frac{1}{2}}$ at each spatial cell center, which in turn provides the inital flux value for the angular derivative term in the equation for $\psi_{m=1}$. During the sweep for $\psi_{m=1}$, the angular outflow flux, $\psi_{m=3/2}$ is calculated in at each spatial cell center, which in turn provides the angular inflow flux for $\psi_{m=2}$. One next performs a sweep for

$\psi_{m=2}$ and then continues in sequence to sweep all of the directions. One peculiarity of the method is that all of the fluxes at the origin must be set to the starting direction flux value at the origin. This follows from the fact that only the starting direction flux points toward the origin. Thus all fluxes at the origin are rigorously equal to the starting direction flux. The starting value can be explicitly imposed upon the forward fluxes ($\mu > 0$) at the origin because they represent incoming fluxes for the first spatial cell. However, this value cannot be explicitly imposed upon the backward fluxes ($\mu < 0$) at the origin because they represent outgoing fluxes for the first spatial cell, and thus are determined by the difference equations for that cell. We can nonetheless explicitly set the backward fluxes at the origin equal to the starting direction flux at the origin if we replace the spatial diamond-difference auxilliary equation for the backward fluxes in the first spatial cell with the explicit definition of the origin fluxes. This means that the average backward fluxes in the first spatial cell must be calculated using only the balance equation together with the angular weighted-diamond auxilliary equation.

2 Diffusion-Synthetic Acceleration

The diffusion-synthetic acceleration process is essentially the same in all S_n geometries, but one must derive the consistent diffusion equation from the S_n equations themselves.

As in the case of slab geometry, this is done via a Galerkin method applied to the fully discretized S_n equations. In particular, one first assumes that

$$\psi_m = \frac{1}{4\pi}\phi + \frac{3}{4\pi}J\mu_m \,, \tag{2}$$

at all spatial points, i.e.,

$$\psi_{i,m} = \frac{1}{4\pi}\phi_i + \frac{3}{4\pi}J_i\mu_m \,, \tag{3}$$

and

$$\psi_{i\pm\frac{1}{2},m} = \frac{1}{4\pi}\phi_{i\pm\frac{1}{2}} + \frac{3}{4\pi}J_{i\pm\frac{1}{2}}\mu_m \,. \tag{4}$$

Next, one substitutes this dependence into the fully discretized S_n equations, and takes the zero'th and first angular moments of these equations using the quadrature formula. The zero'th moment yields the balance equation:

$$A_{i+\frac{1}{2}}J_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}J_{i-\frac{1}{2}} + \sigma_{a,i}\phi_i V_i = q_i V_i \,. \tag{5}$$

The first moment yields (after considerable algebra and manipulation) a form of Fick's law :

$$J_i = -\frac{1}{3\sigma_{t,i}}\left(\phi_{i+\frac{1}{2}} - \phi_{i-\frac{1}{2}}\right)\frac{A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}}{2V_i} \,. \tag{6}$$

Expanding the geometric quantity on the right side of Eq. (6), we get

$$\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2V_i} = \frac{r_i^2 + \Delta r_i^2}{r_i^2 + \delta r_i^2/12}\frac{1}{\Delta r_i} \,. \tag{7}$$

Clearly, this expression limits to $\frac{1}{\Delta r_i}$ as $r \to \infty$ and as $\Delta r_i \to 0$, which is necessary for convergence to the analytical form of Fick's law. To obtain a diffusion equation, we proceed as in the slab-geometry case and sum the balance equation over two cells:

$$A_{i+\frac{3}{2}} J_{i+\frac{3}{2}} - A_{i-\frac{1}{2}} J_{i-\frac{1}{2}} + \sigma_{a,i+1}\phi_{i+1}V_{i+1} + \sigma_{a,i}\phi_i V_i = q_{i+1}V_{i+1} + q_i V_i \,. \tag{8}$$

At this point we must make an approximation that is not necessary in slab geometry. In particular, We assume that:

$$A_{i+\frac{3}{2}} J_{i+\frac{3}{2}} - A_{i-\frac{1}{2}} J_{i-\frac{1}{2}} = 2\left(A_{i+1}J_{i+1} - A_i J_i\right) \,. \tag{9}$$

This is not an approximation if one calculates A_i as a current-weighted average:

$$A_i = \frac{A_{i+\frac{1}{2}} J_{i+\frac{1}{2}} + A_{i-\frac{1}{2}} J_{i-\frac{1}{2}}}{J_{i+\frac{1}{2}} + J_{i-\frac{1}{2}}} \,. \tag{10}$$

However, this would result in a non-linear equation, so instead we define the mid-point area as follows:

$$A_i = \frac{1}{2}\left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right) \,. \tag{11}$$

Substituting from Eq. (9) into Eq. (5), and eliminating the currents using Eq. (6), we get a three-point diffusion discretization:

$$-A_{i+1}\frac{1}{3\sigma_{t,i+1}}\left(\phi_{i+\frac{3}{2}} - \phi_{i+\frac{1}{2}}\right)\frac{A_{i+\frac{3}{2}} + A_{i+\frac{1}{2}}}{2V_{i+1}} + A_i \frac{1}{3\sigma_{t,i}}\left(\phi_{i+\frac{1}{2}} - \phi_{i-\frac{1}{2}}\right)\frac{A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}}{2V_i} +$$
$$\frac{1}{4}\sigma_{a,i+1}V_{i+1}\left(\phi_{i+\frac{3}{2}} + \phi_{i+\frac{1}{2}}\right) + \frac{1}{4}\sigma_{a,i}V_i\left(\phi_{i+\frac{1}{2}} + \phi_{i-\frac{1}{2}}\right) = \frac{1}{2}q_{i+1}V_{i+1} + \frac{1}{2}q_i V_i \,. \tag{12}$$

Appropriate Marshak conditions are used to close the system at the boundaries. For instance, the boundary condition at the center of the sphere is reflective, so $J_{\frac{1}{2}} = 0$. The equation for the $\phi_{\frac{1}{2}}$ represents a balance over the interval $[0, r_1]$. This equation takes the following form:

$$-A_1 \frac{1}{3\sigma_{t,1}} \left(\phi_{\frac{3}{2}} - \phi_{\frac{1}{2}}\right) \frac{A_{\frac{3}{2}} + A_{\frac{1}{2}}}{2V_1} + \frac{1}{4}\sigma_{a,1} V_1 \left(\phi_{\frac{3}{2}} + \phi_{\frac{1}{2}}\right) = \frac{1}{2} q_1 V_1 \,. \tag{13}$$

Let I denote the total number of spatial cells. The equation for $\phi_{I+\frac{1}{2}}$ represents a balance over the interval, $[r_I, r_{I+\frac{1}{2}}]$. Assuming a vacuum condition at the outer surface of the sphere, the equation for ϕ_I is

$$\frac{1}{2} A_{I+\frac{1}{2}} \phi_{I+\frac{1}{2}} + A_I \frac{1}{3\sigma_{t,I}} \left(\phi_{I+\frac{1}{2}} - \phi_{I-\frac{1}{2}}\right) \frac{A_{I+\frac{1}{2}} + A_{I-\frac{1}{2}}}{2V_I} + \frac{1}{4}\sigma_{a,I} V_I \left(\phi_{I+\frac{1}{2}} + \phi_{I-\frac{1}{2}}\right) = \frac{1}{2} q_I V_I \,. \tag{14}$$